Modelling Geomorphological Systems

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Chapter 10

Modelling erosion on hillslopes

L. J. LANE and E. D. SHIRLEY
Southwest Rangeland Watershed Research Center, Agricultural Research Service, Tucson
and
V. P. SINGH
Department of Civil Engineering, Louisiana State University

10.1 INTRODUCTION

Surface runoff on upland areas such as hillslopes is often accompanied by soil erosion. Soil particles may be detached when the impact of raindrops exceeds the soil's ability to withstand the impulse at the soil surface. Detachment may also occur when shear stresses caused by flowing water exceed the soil's ability to resist these erosive forces. Vegetation as canopy and ground cover, and other surface cover such as gravel and rock fragments, protect the soil surface from direct raindrop impact, and also provide hydraulic resistance, reducing the shear stresses acting on the soil. Plant roots, incorporated plant residue, and minerals increasing cohesion tend to protect the soil by reducing the rate of soil particle detachment by flowing water and raindrop impact.

Once detachment has occurred, sediment particles are transported by raindrop splash and by overland flow. Conditions which limit raindrop detachment limit the sediment supply available for transport by splash and flow mechanisms. Vegetative canopies intercept splashed sediment particles and limit sediment transport by splash. The rate of sediment transport by overland flow is influenced by the factors controlling the amount of sediment available for transport, the sediment supply, and by hydraulic processes occurring in overland flow such as raindrop impacts, depth of flow, velocity, and accelerations due to microtopographic flow patterns. Obviously, the steepness, shape, and length of slopes affect both flow patterns and the resulting sediment transport capacity of the flowing water.

After sediment particles are detached from soil areas above, between, and near locations of small flow concentrations, they may enter the flow concentration areas for subsequent transport downslope by hydraulic processes.

Throughout the remainder of the chapter, the flow concentration areas are called rills, and the areas between the rills are called interrill areas. Together, these interrill and rill areas make up the overland flow surface. Sediment particles detached in the interrill areas move to the rills by the processes of splash as the result of raindrop impact, and by suspension and saltation in overland flow. The rate of delivery of water and sediment to the rills affects the rates of sediment detachment, transport, and deposition in the rills.

Deposition occurs in overland flow when sediment particles come to rest on the soil surface, which occurs when sediment load in the flow exceeds the flow's capacity to transport the sediment. The rate of deposition is determined by both flow characteristics affecting energy, momentum, and turbulence and sediment particle characteristics, including particle interactions, affecting fall or settling velocity.

Thus, the processes controlling sediment detachment, transport, and deposition on the hillslope scale, lumped under the term erosion processes, are complex and interactive. This complexity leads to the need for upland erosion models as tools in resource management. Erosion models and observations are superior to observations alone, because simultaneous observation and measurement of all the processes controlling surface runoff and erosion are beyond the current and foreseeably available technology. Moreover, observations and measurements are particularly difficult, due to the small temporal and spatial scales necessary during a runoff and erosion event. Quite often, after the fact observations are the best that can be obtained. Almost as often, these post event observations reveal little of the actual mechanisms causing the erosion.

Ideally, an erosion model should represent the essential mechanisms controlling erosion, and the model parameters should be directly related to measurable physical properties. However, under real conditions, all models are more or less incorrect, because all models are abstractions and simplifications of the actual physical processes. Moreover, model parameters are often impossible or difficult to directly measure, and thus are always, to some extent, data based rather than predetermined. These real-world problems with mathematical models of overland flow and erosion have resulted in three main types of models: those that are primarily empirically based; those that are partially conceptually based and partially empirically based; and those that are partially process based or physically based and partially empirically based. As will be illustrated in subsequent discussions, these three main types of models are typified by the Universal Soil Loss Equation (USLE) as described by Wischmeier and Smith (1978), by the unit sediment graph (i.e. Rendon-Herrero, 1978; Williams, 1978), and by coupled overland flow-erosion equations based on the concepts of kinematic flow and separable rill and interrill erosion processes (i.e. Foster, 1982 and his example model listed on pp. 370-372). Some examples of the three types of models are shown in Table 10.1.

10.1.1 Developments Resulting in the USLE

Recently, Meyer (1984) and Nyhan and Lane (1986) summarized the evolution of the USLE, and the latter divided its development into four historical periods. The first period (1890s-1940) was described as a period wherein a basic understanding of most of the factor affecting erosion was obtained in a qualitative sense (Cook, 1936). This period include the rainfall studies of Laws (1940) and the analyses of the action of raindrops in erosion reported by Ellison (1947).

Table 10.1. Some examples of empirical and conceptual models

Model Type	Model	Author
Empirical	Musgrave Equation Universal Soil Loss Equation (USLE) Modified Universal Soil Loss Equation (MUSLE) Sediment Delivery Ratio Method Dendy-Boltan Method Flaxman Method Pacific Southwest Interagency Committee (PSIAC) Method Sediment Rating Curve Runoff-Sediment Yield Relation	Musgrave (1947) Wischmeier and Smith (1978) Williams (1975) Renfro (1975) Dendy and Boltan (1976) Flaxman (1972) Pacific Southwest Interagency Interagency Committee (1968) Campbell and Bauder (1940) Rendon-Herrero (1974), Singh, Baniukiwicz and Chen (1982)
Conceptual	Sediment Concentration Graph Unit Sediment Graph Instantaneous Unit Sediment Graph Discrete Dynamic Models Renard-Laursen Model Sediment Routing Model Muskingum Sediment Routing Model	Johnson (1943) Rendon-Herrero (1978) Williams (1978) Sharma and Dickinson (1979) Renard and Laursen (1975) Williams and Hann (1978) Singh and Quiroga (1986)
Physically based	Quasi-Steady State Erosion Kinematic Wave Models Continuum Mechanics Model	Foster. Meyer and Onstad (1977) Hjelmfelt, Piest and Saxton (1975), Shirley and Lane (1978), Singh and Regl (1983) Prasad and Singh (1982)

During the period 1940-1954, work in the Corn Belt of the United States resulted in a soil loss estimation procedure incorporating the influence of slope length and steepness (Zingg, 1940), conservation practices (Smith, 1941; Smith and Whitt, 1947), and soil and management factors (Browning *et al,* 1947). In 1946, a national committee reappraised the Corn Belt factor values, included a rainfall factor, and produced the resulting Musgrave equation (Musgrave, 1947).

During the period 1954-1965, the USLE was developed by the United States Department of Agriculture (USDA), Agricultural Research Service in cooperation with the USDA-Soil Conservation Service and state agricultural experiment stations. Plot data from natural storms and from rainfall simulator studies formed the USLE data base. During the 1965-1978 period, additional data and experimental results were incorporated, resulting in the current USLE (Wischmeier and Smith, 1978).

The USLE in equation form is:

$$A = RKLSCP \tag{1}$$

where:

A = the computed soil loss per unit area (tons per acre-yr).
R = the rainfall and runoff factor (hundreds of ft-tons-in per acre-hr-yr),
K = the soil erodibility factor (tons-acre-hr per hundreds of acre-ft-tons-in),
LS = the slope length-steepness factor (1.0 on uniform 72.6 ft slope at 9 per cent steepness),
C = the cover-management factor (1.0 for tilled, continuous fallow), and
P = The supporting practices factor (1.0 for up and down hill tillage, etc.).

The original USLE was presented in English units, hence their usage here. The unit plot (where *LS, C* and *P* are all equal to 1.0) is defined as a clean tilled, up and down slope, 72.6 ft long plot with a uniform 9 per cent slope. For slope lengths of 30 to 300 ft and steepness from 3 to 18 per cent, the *LS* factor ranges from a low of about 0-2 to a high of about 6. Values of the *C* factor range from a low of about 0.003, for near complete grass cover, to 1.0 for the unit plot. Values of the *P* factor range from 0.5 for contouring to 1.0 for the unit plot. Values of the *R* factor range from under 20 to over 550 in the continental United States, with some values outside these limits in other parts of the world representing greater climatic extremes. Wischmeier and Smith (1978, pp. 8-11) list values of the soil erodibility factor, *K*, ranging from 0.03 to 0.69, with most values in the range 0.2 to 0.4. With appropriate values of the above factors, the USLE is intended to predict the long-term average annual soil loss from uniform slopes, or from nonuniform slopes without deposition (Foster and Wischmeier, 1974).

That the USLE remains the most widely used tool in predicting upland erosion supports the description of upland erosion processes as complex and interactive. The state of the art is such that more conceptual and processes based erosion prediction equations for practical applications are just emerging, and do not yet have wide usage.

10.1.2 Development of Conceptual Models

The conceptual models lie somewhere between empirically and physically based models, and are based on spatially lumped forms of continuity equations for water and sediment and some other empirical relationships. Although highly simplified, they do attempt to model the sediment yield, or the components thereof, in a logical manner. To summarize, conceptual models of sediment are analogous in approach to chose of surface runoff, and hence, embody the concepts of the unit hydrograph (UH) theory. Rendon-Herrero (1974, 1978) was probably the first to have extended this theory to derive a unit sediment graph (USG) for a small watershed. The sediment load considered in the USG is the wash load only. Rendon-Herrero (1974) expressed the following to define the USG:

> A form of a unit sediment graph was indeed developed whose standard unit was 1.0 ton (910kg) for a given duration, distributed over the watershed, analogous in unit-hydrograph analysis to 1.00 in. (25 mm) of excess (effective) rainfall over the same area.

In light of this definition, the USG and UH are similar in their derivations. To discuss the derivation of the USG by Rendon-Herrero, the following steps are outlined.

1. Select an isolated rainfall-runoff event of a desired duration in accordance with the requirement of the UH for which the sediment concentration graph C is known.

2. Separate the baseflow Q_b from the runoff hydrograph Q_T using a standard, hydrograph separation technique to obtain the direct runoff hydrograph Q,

$$Q(t) = Q_T(t) - Q_b(t) \tag{2}$$

3. Using the same baseflow separation technique, separate out the sediment concentration due to baseflow. It should be noted that Rendon-Herrero assumed that the maxima of runoff and sediment concentration occurred at the same time.
4. Compute sediment discharge Q_s due to direct runoff by noting that sediment discharge is the product of water discharge and sediment concentration,

$$Q_S = Q_T C_T - Q_b C_b \tag{3}$$

5. Compute the volume of direct runoff, which is the area under the direct runoff hydrograph.

$$V_Q = \int_0^\infty Q(t)\mathrm{d}t \tag{4}$$

6. Compute the sediment yield, which is the area under the sediment graph due to direct runoff.

$$V_S = \int_0^\infty Q_S \mathrm{d}t \tag{5}$$

7. Divide the ordinates of the sediment graph by the sediment yield to obtain ordinates of the USG, H_s,

$$H_S = \frac{Q_S}{V_S} \tag{6}$$

The USG varies somewhat with the intensity of the effective rainfall. It can be used to generate a sediment graph for a given storm if the wash load produced by that storm is known. A relationship between V_s and V_Q was proposed. Using this relation, V_s can be determined. Therefore, Q_s can be determined by multiplying H_s with V_s. It must be noted that the duration of the USG chosen to determine Q, must be the same as that of the effective rainfall generating V_Q. This USG method was tested on a small wash loadproducing watershed, Bixler Run Watershed, near Loysville, Pennsylvania.

Rendon-Herrero (1974) proposed the use of the so-called 'series' graph to determine the sediment hydrograph. This method has the advantage that the duration of the effective rainfall is neglected altogether, but requires construction of the series graphs beforehand. Thus, this method cannot be extended to ungauged basins. Williams (1978) and Singh *et al* (1982), among others, have used the USG to model watershed sediment yield.

10.1.3 Development of Physically Based Erosion Models

Fundamental erosion mechanics were of interest to scientists and engineers as early as 1936 (Cook, 1936), and were described in terms of subprocesses by Ellison (1947). Negev (1967) included an erosion component in the Stanford

Watershed Model (Crawford and Lindsley, 1962). Meyer and Wischmeier (1969) presented relationships for the major erosion subprocesses, and incorporated them in a model of overland flow erosion, which formed the conceptual basis of most subsequent erosion modelling efforts.

Foster and Meyer (1972) published a paper on a closed-form soil erosion equation for overland flow, which demonstrated the ability of models in this class to provide insight into the spatial variability of erosion on hillslopes and into the separable interrill and rill erosion processes. This analysis assumed steady state conditions, and emphasized spatially variable processes. However, it set the stage for subsequent analyses of spatially varying and unsteady overland flow and erosion.

Hjelmfelt, Piest, and Saxton (1975) solved the coupled partial differential equations for overland flow with interrill and rill erosion and constant and uniform rainfall excess. However, they solved them only for the rising and steady state portions of the overland flow hydrograph. Shirley and Lane (1978) solved the equations for constant and uniform rainfall excess of finite duration over the entire overland flow hydrograph using the method of characteristics, and then integrated the equations to produce a sediment yield equation for the entire runoff hydrograph. Singh and Prasad (1982) advanced the models by formulating the partial differential equations for overland flow and erosion on an infiltrating plane, and then presented analytic solutions, by the method of characteristics, for the special case of constant and uniform rainfall and infiltration, or constant and uniform rainfall excess, on a sloping plane. Also, see Singh (1983) for a more complete description of the methods of solution. Solution domains and analytic solutions of the overland flow and interrill and rill erosion equations for the special case of constant and uniform rainfall excess on a plane are given in the Appendix. These solutions can be examined in analytic form to illustrate changes in sediment concentration in time and space for the case of unsteady and spatially variable overland flow.

Subsequent investigators examined various approximations to the analytic solutions described above (e.g. Rose *et al.*, 1983a), and their fit to measured data (Rose *et al.,* 1983b). Lane and Shirley (1982) also discussed the fit of the coupled overland flow and interrill and rill erosion equations to time varying runoff and sediment concentration data from plots and a small watershed on the Walnut Gulch Experimental Watershed in southeastern Arizona, USA. Blau (1986) examined the parameter identifiability of the overland flowerosion model for the special case of constant and uniform rainfall excess on a plane. He concluded that, because of parameter interactions in the model, parameter values were difficult to obtain by least squares optimization using measured data.

As indicated above, the research reported on the more physically based overland flowerosion equations are representative of mathematical derivations and manipulations, or of efforts to determine parameter values by fitting the models to measured data.

10.1.4 Scope and Limitations

Subsequent discussions will be primarily limited to erosion processes occurring in overland flow on plots and hillslopes. Although some of the more major assumptions and approximations used in deriving solutions to the governing equations are described, the main emphasis is on their solutions after the simplifying assumptions and the mathematical and practical significance of the approximating equations and their solutions.

10.1.5 Purpose

The first purpose of this chapter is to describe the evolution and status of erosion models for hillslopes based upon the kinematic wave equations for overland flow, and on the interrill and rill terms for erosion. The second purpose is to examine a particular erosion model for which analytic solutions can be obtained, and then to discuss the mathematical properties and implications of the solutions as they relate to experimental design and interpretation of experimental data.

10.2 OVERLAND FLOW AND EROSION EQUATIONS

The development of improved erosion equations for overland flow is based upon prior development of improved flow equations. That is, the development of methodology for simulation of unsteady and spatially varying overland flow made the subsequent simulation of interrill and rill erosion possible.

10.2.1 The Shallow Water Equations

Unsteady and spatially varying and one-dimensional flow per unit width on a plane was described by Kibler and Woolhiser (1970) using the following equations:

$$\frac{\partial h}{\partial t} + \frac{u^{\partial} h}{\partial x} + \frac{h^{\partial} u}{\partial x} = R \tag{7}$$

and

$$\frac{\partial u}{\partial t} + \frac{u \partial h}{\partial x} + \frac{g \partial h}{\partial x} = g(S_o - S_f) - (R/h)(u - v) \tag{8}$$

where

h = local depth of flow (dimension of length, L),
u = local mean velocity (L/T),
t = time (T),
x = distance in the direction of flow *(L),*
R = lateral inflow rate per unit area (L/T),
g = acceleration of gravity (L/T^2),
S_o = slope of the plane.
S_f = friction slope, and
v = velocity component of lateral inflow in the direction of flow (L/T).

Equation 7 is the continuity of mass equation, and equation 8 is the one-dimensional momentum equation. In general, equations 7 and 8 must be solved numerically. Modelling real overland flow with one-dimensional equations represents significant abstractions and simplifications. Real overland flow occurs in complex mixes of sheet flow and small concentrated flow areas. The routes of concentrated flow are often determined by irregular microtopographic features which vary in the downstream direction (x) and in the lateral direction (y). Definitive analyses of the influences of such simplifications upon hydraulic and erosion parameters are nonexistent.

The lateral inflow, R, in equations 7 and 8, is, in reality, a complex function of time and space representing all the variations in rainfall input and in infiltration. It is often represented as the positive difference between instantaneous rates of rainfall and infiltration, or as zero if infiltration rate exceeds rainfall rate. This positive difference is called rainfall excess. In solving equations 7 and 8, a typical assumption is that a block of rainfall can be divided into infiltration and rainfall excess. Rainfall excess is then routed as if the surface were impervious, which is a significant simplification (Smith and Woolhiser, 1971) Moreover, infiltration is usually assumed to be uniform over the overland flow surface, while in reality, infiltration rates vary significantly in space. The assumption of spatially uniform infiltration, and thus rainfall excess, is a serious limitation in most current modelling approaches, and may preclude accurate prediction of overland flow under many natural conditions (i.e. Lane and Woolhiser, 1977).

The velocity component, v, in equation 8, is almost always assumed to be zero. This assumption may be reasonable on natural overland flow surfaces for distances on the order of a meter or larger. The validity of this assumption has not been tested on a smaller scale, on the order of a centimeter or so, and v may be quite significant in raindrop impact and sediment detachment and transport processes at this scale.

10.2.2 The Kinematic Wave Equations

If all terms in the momentum equation, equation 8, axe assumed to be small in comparison with the $g(S_o - S_f)$ term and can be neglected, then the shallow water equations become the kinematic wave equations. The kinematic wave equations for overland flow per unit width on a plane are:

$$\frac{\partial h}{\partial t} + \frac{\partial q}{\partial x} = R \tag{9}$$

and

$$q = Kh_m \tag{10}$$

where:

q = the local runoff rate per unit width ($L^2\ T^{-1}$),
K = the stage-discharge coefficient ($L^{m-1}\ T^{-1}$), and
m = the exponent dependent upon the friction law assumed.

The exponent m is 3/2 for the Chezy equation and 5/3 for the Manning equation. Throughout the remainder of this chapter, the Chezy form will be used so that m = 3/2 and

$$K = C\sqrt{S} \tag{11}$$

where C is the Chezy resistance coefficient ($L^{1/2}T^{-1}$), and S is the slope of the plane surface.

Lighthill and Whitham (1955) introduced the kinematic wave theory for flood routing in rivers and for overland flow. Iwagaki (1955) used the kinematic assumptions and a method of characteristics for unsteady flow in rivers. Henderson and Wooding (1964) used the kinematic wave equations for steady rain of finite duration and for flow over a sloping plane. Woolhiser and Liggett

(1967) showed that solutions to the kinematic wave equations are a good approximation to the solutions to the shallow water equations, provided the kinematic flow number is larger than about 20. It is important to note that this refers to the accuracy with which the kinematic wave solutions approximate solutions to the shallow water equations for sheet flow on a plane. The kinematic flow number says nothing about how well the shallow water equations, with one-dimensional flow and spatially uniform parameters, approximate overland flow on natural surfaces.

10.2.3 Equations for Erosion by Overland Flow

The sediment continuity equation, with the kinematic assumptions, is quite similar to the water continuity equation on the left hand side. The right hand side of the sediment continuity equation is commonly separated into an interrill erosion term, E_I, and the rill erosion term, E_R. With these assumptions, the continuity equation for sediment is:

$$\frac{\partial(ch)}{\partial t} + \frac{\partial(cq)}{\partial x} = E_I + E_R \tag{12}$$

where:

c = sediment concentration (M L^{-3}),
E_I = interrill erosion rate per unit area per unit time (M L^{-2} T^{-1}), and
E_R = net rill erosion (or deposition) rate (M L^{-2} T^{-1}),

and the other variables are as described earlier. The procedure is to solve the flow equations first, and then solve equation 12 for sediment concentration. Total sediment yield for a storm, V_s, is then found by integrating the product cq over the period of runoff.

The interrill term, E_I

The rate of interrill erosion is a function of the rate of detachment by raindrop impact and the rate of transport from the point of detachment to a rill.

As discussed in the introduction, interrill erosion is, by definition, caused by raindrop detachment and the rate of transport in the shallow interrill flow. On steep slopes, the rate of detachment by raindrop impact limits interrill erosion, whereas transport capacity in interrill flow limits the rate of delivery on flat slopes (Foster, Meyer, and Onstad, 1977). These authors, and others, document the dependence of interrill erosion on soil characteristics, slope steepness, and canopy and ground cover. In equation form, this can be expressed as

$$E_I = f(I, S, C, Soil) \tag{13}$$

where I, S, and C are rainfall intensity, slope of the land surface, and cover effects, respectively. *Soil* refers to the soil characteristics, primary particle-size distribution, type and amount of clay and crusting, and land use influencing soil properties, such as density and aggregation, which affect raindrop detachment and shallow flow. Following are some selected interrill erosion terms.

A simple functional form incorporating rainfall intensity, I, as a measure of the erosivity of raindrop impact is

$$E_I = aI^2 \tag{14}$$

where *a* is a coefficient to be determined experimentally. If the production of rainfall excess is related to *I*, and the transport capacity of shallow flow is, in turn, related to the rainfall excess, then a simple interrill erosion equation is

$$E_I = bR \tag{15}$$

where *b* is a coefficient to be determined. If the rate of detachment is related to the rainfall intensity squared, and the flow transport capacity is related to the ratio of rainfall excess to rainfall intensity, then a simple form of the interrill erosion equation is

$$E_I = cI^2(R/I) = cIR \tag{16}$$

where *c* is a coefficient to be determined. Additional information on a number of expressions for interrill erosion rates is given by Foster *et al.* (1982).

The rill term, E_R

There are two common ways of expressing soil detachment in rills, and one common way of expressing the rate of sediment deposition in rills. While more expressions or functional forms for detachment and deposition are available, the following material is indicative of modern erosion science.

If the rate of soil detachment in a rill is assumed to be a function of the shear stress in excess of a critical shear stress, then the following equation describes the rate of rill erosion:

$$Er = d(\tau - \tau_c)^e \tag{17}$$

where *d* is a coefficient to be determined, τ is the average shear stress in the cross-section, τ_c, is a critical shear stress that must be exceeded to initiate soil detachment, and *e* is an exponent to be determined.

A second major class of rill erosion equations results when one assumes the rate of rill erosion is proportional to the amount the flow transport capacity, T_c, is in excess of the existing sediment load, *cq*. These equations are of the form

$$Er = f(T_c - cq) \tag{18}$$

where *f* is a coefficient to be determined, and the other variables are as described above.

Two issues are involved in selecting a rill erosion equation of the type discussed here. The first issue is whether or not one assumes an interaction among rill erosion, sediment load, and transport capacity. Meyer and Wischmeier (1969) neglected the interaction, and their model represents the first major class of rill erosion models. Foster and Meyer (1972) assumed an interaction, and their model represents the second major class of rill erosion models. The second issue is whether or not one assumes a critical shear stress in determining the rate of detachment, as in equation 17, or the transport capacity used in equation 18.

In the event that more sediment is delivered to the channel segment from upstream and from lateral inflow than its transport capacity, then sediment deposition will occur in the rill segment at a rate proportional to the deficit in transport capacity. This means that equation 18 can describe the rate of deposition if the coefficient f is a deposition coefficient. The deposition coefficient is primarily a function of particle characteristics. and is often calculated as a function of the particle fall velocity and the steady-state discharge rate (Foster, 1982).

10.2.4 Numerical Solutions

As stated earlier, equations 7 and 8 are solved numerically. Finite difference techniques are usually used (i.e. see Kibler and Woolhiser, 1970). If R, in equation 9, varies in space and time, then equations 9 and 10 must be solved numerically. If R in equation 10 varies, or if E_I and E_R in equation 12 are complex functions, then equation 12 must be solved numerically. The advantage of numerical techniques in solving the above equations is that one need not make as many assumptions as is required for analytic solutions, and the rainfall excess term can vary in time and space.

The disadvantages of numerical techniques, compared with analytic solutions, is that the former usually require much more computer time, the solutions are approximations of the real solutions, and the mathematics required for sensitivity analysis, limits, and other manipulations may be unavailable or very complex and difficult.

10.2.5 Analytic Solutions

Equations 9 and 10 can be solved analytically (by the method of characteristics) if R *is* uniform over the plane, and the temporal variation in R is described by a series of step functions. However, to obtain an analytic solution for equation 12, R in equation 9 must be uniform and constant for a finite or infinite duration. Equations 9 and 10 must be solved first to substitute into equation 12. Also, the form of T_c, in equation 18, should be simple, for example, a linear function of q, to obtain an analytic solution.
As stated earlier, the disadvantages of analytic solutions, in comparison with numerical solutions, are that they usually require much more restrictive and simplifying assumptions. The main advantages of analytic solutions include the case with which they can be implemented on a computer, the speed with which they can be evaluated, the simplicity of sensitivity analysis, and the case with which one can examine limits and ocher mathematical properties of the solutions.

10.3 SIMPLIFIED EQUATIONS WITH ANALYTIC SOLUTIONS

In this section, specific assumptions and simplifications are made to allow the derivation of analytic solutions for overland flow on a plane, and for interrill and rill erosion with overland flow. Analytic solutions to the runoff and erosion equations are used to illustrate field data needed for estimation of parameter values and for interpretation of processes controlling erosion.

10.3.1 The Basic Assumptions

In addition to the assumptions necessary for derivation of the one-dimensional shallow water equations and their approximating kinematic wave equations, specific assumptions are required for the erosion equations to have an analytic solution. In equation form, the assumptions are:

$$\frac{\partial(ch)}{\partial t}+\frac{\partial(cq)}{\partial x}=K_I R+K_R((B/K)q-cq) \tag{19}$$

with initial and boundary conditions as

$$c(0,x)=K_I \tag{20}$$

and

$$c(t,0)=K_I \tag{21}$$

We also assume a pulse input of the form

$$R(t)=\begin{cases} R & \text{for} \quad 0<t<t_* \\ 0 & \text{otherwise} \end{cases} \tag{22}$$

for the rainfall excess.

The first term on the right hand side of equation 19 means that we assume

$$E_I=K_I R \tag{23}$$

and the second term on the right hand side of equation 19 means that we assume

$$E_R=K_R(T_c-cq) \tag{24}$$

with $T_c = Bh^{3/2}$. Since $q = Kh^{3/2}$, we can write $T_c = (B/K)q$, and the result is the second term on the right hand side of equation 19.

Equations 10 and 21 mean that the initial concentration is K_I and, furthermore, that the concentration at the upstream boundary remains equal to K_I throughout the runoff hydrograph. These results can also be seen by taking limits of the equations presented in the Appendix. These assumptions and results are very significant in designing field experiments and in interpreting the resulting data.

10.3.2 Implications

The limit of the concentration as t approaches zero is:

$$C_0=K_I \tag{25}$$

as the initial concentration. The limit of $c(t,x)$, for fixed x and as t approaches infinity, is C_f, and is given by

$$C_f=B/K+(K_I-B/K)\exp(-K_R x) \tag{26}$$

Finally, Shirley and Lane (1978) showed that the mean concentration, C_b, over the entire hydrograph is

$$C_b = Qs/Q = B/K + (K_I - B/K)(1-\exp(-K_R x))/K_R x \tag{27}$$

If $B/K > K_I$, then $C_o < C_b < C_f$ and $c(t,x)$ for fixed x is a non-decreasing function of t. It can also be shown for fixed t that if $B/K > K_I$, *then* $c(t,x)$ is a non-decreasing function *of* x. These two non-decreasing functions mean (in the context of this particular model) that if $B/K > K_I$, then there is more transport capacity in the rills than is being satisfied by sediment input from the internal areas. As a result, rill erosion occurs at all times and at all positions on the plane. In terms of sediment concentration graphs measured in the field, measured concentrations would tend to start at K_I *near* $t = 0$, and increase throughout the duration of runoff, assuming, of course, that the model is a good representation of reality.

If $B/K < K_I$, then the opposite is true. Under these conditions, $c(t,x)$ for fixed x would be non-increasing, or tend to decrease with increasing t. Also, $c(t,x)$ would be non-increasing with x and a fixed t. Again, if the model is correct, then measured concentrations would tend to start at K_I near $t = 0$, and decrease throughout the duration of runoff. *If* $B/K = K_I$, then transport capacity and existing sediment load are in equilibrium, so $C_o = C_f = C_b$, and, in fact, $c(t,x) = KI$ for all x and t.

The implications of these results for plot and hillslope studies are that sediment concentration should be measured throughout the duration of runoff, and that analysis of data, using this model for parameter identification, should concentrate on events with nearly constant rainfall intensity and nearly saturated initial soil water content. The last two conditions will tend to make rainfall excess nearly constant, as assumed in the analysis. Fortunately, these conditions can nearly be met in rainfall simulator studies if data from runs where the initial soil water content is near saturation and the infiltration rate is nearly a constant are obtained for analysis.

Therefore, as a first approximation, one can examine the shape of the sediment concentration *vs*. time curve from a particular event on an experimental plot, and infer whether transport capacity in the rills ($B/K < K_I$) or detachment rate ($B/K > K_I$) in the rills is limiting sediment yield.

10.4 DISCUSSION

Although the Universal Soil Loss Equation remains the most often used model for predicting erosion on upland areas, more physically based models are emerging, and may become practical tools in the near future (i.e. see Rawls and Foster, 1986). As these new models emerge, they will probably be based upon unsteady and nonuniform overland flow modelled with the kinematic wave equations. Moreover, interrill and rill erosion processes will probably be explicitly represented in the partial differential equation used to describe erosion and overland flow.

The implications for plot and hillslope studies are that more, and more intensive, data need to be collected throughout the duration of runoff events, and at various positions on the slope. Only then can we begin to quantify unsteady and spatially varying overland flow and erosion processes.

APPENDIX

Summary of the Solution Regions and Solutions for the Overland Flow Equations in the $t - x$ Plane

Recall that the kinematic wave equations are:

$$\frac{\partial h}{\partial t} + \frac{\partial q}{\partial x} = R \tag{A1}$$

and

$$q = Kh^m \tag{A2}$$

where the variables are as defined previously in the text.

1. Domains in the $t - x$ Plane for Solutions of the Kinematic Overland Flow Equations

Solutions for the overland flow equations require that the positive quadrant of the $t - x$ plane be divided into four regions. The regions listed below are also presented in Figure 10.4.

a. *Domain of Flow Establishment*. This region of the $t - x$ plane represents time from zero until cessation of rainfall excess at time t_* and distance down the plane such that steady state has not been reached.

$$\begin{aligned} &0 \le t \le t_* \\ &x \ge KR^{m-1}t^m \end{aligned} \tag{A3}$$

b. *Domain of Established Flow*. This region of the plane represents time from zero until cessation of rainfall excess and distance down the plane such that steady state has been reached:

$$\begin{aligned} &0 \le t \le t_* \\ &0 \le x \le KR^{m-1}t^m \end{aligned} \tag{A4}$$

c. *Domain of Prerecession*. This region of the plane represents time after cessation of rainfall excess and before depth of low starts receding:

$$\begin{aligned} &t \ge t_* \\ &x \ge K(1-m)R^{m-1}t_*^m + Km(Rt_*)^{m-1}t \end{aligned} \tag{A5}$$

d. *Domain of Recession*. This region of the plane represents time after the cessation of rainfall excess and depth of flow is receding:

$$\begin{aligned} &t \ge t_* \\ &0 \le x \le K(1-m)R^{m-1}t_*^m + Km(Rt_*)^{m-1}t \end{aligned} \tag{A6}$$

Solutions in the Regions

a. *Domain of Flow Establishment*. In this region, the flow is unsteady but uniform:

$$h(t,x) = Rt \tag{A7}$$

b. *Domain of Established Flow*. In this region, the flow is steady but not uniform:

$$h(t,x) = (Rx/K)^{1/m} \tag{A8}$$

c. *Domain of Prerecession*. In this region the flow is steady and uniform:

$$h(t,x) = Rt_* \tag{A9}$$

d. *Domain of Recession*. In this region, the flow is unsteady and not uniform:

$$h(t,x) = f_t^{-1}(Rx/K) \tag{A10}$$

where

$$f_t(u) = u^m + Rmu^{m-1}(t - t_*) \tag{A11}$$

The solutions described above are also shown in Figure 10.1.

Summary of Solution Regions and Solutions for the Sediment Concentration Equations in the *t – x* Plane

Recall that the erosion equations are:

$$\frac{\partial(ch)}{\partial t} + \frac{\partial(cq)}{\partial x} = E_I + E_R \tag{A12}$$

with

$$E_I = K_I R \tag{A13}$$

and

$$E_R = K_R(Bh^n - cq) \tag{A14}$$

where the variables are defined previously in the text.

1. Domains in the *t – x* Plane for Solutions of the Sediment Concentration Equations

Solutions for the concentration equations require that the positive quadrant of the *t – x* plane be divided into seven regions. The regions listed below are also shown in Figure 10.2.

a. *Domain 1*. This region of the plane represents time from zero until cessation of rainfall excess and distance down the plane such that concentration and flow have not reached steady state:

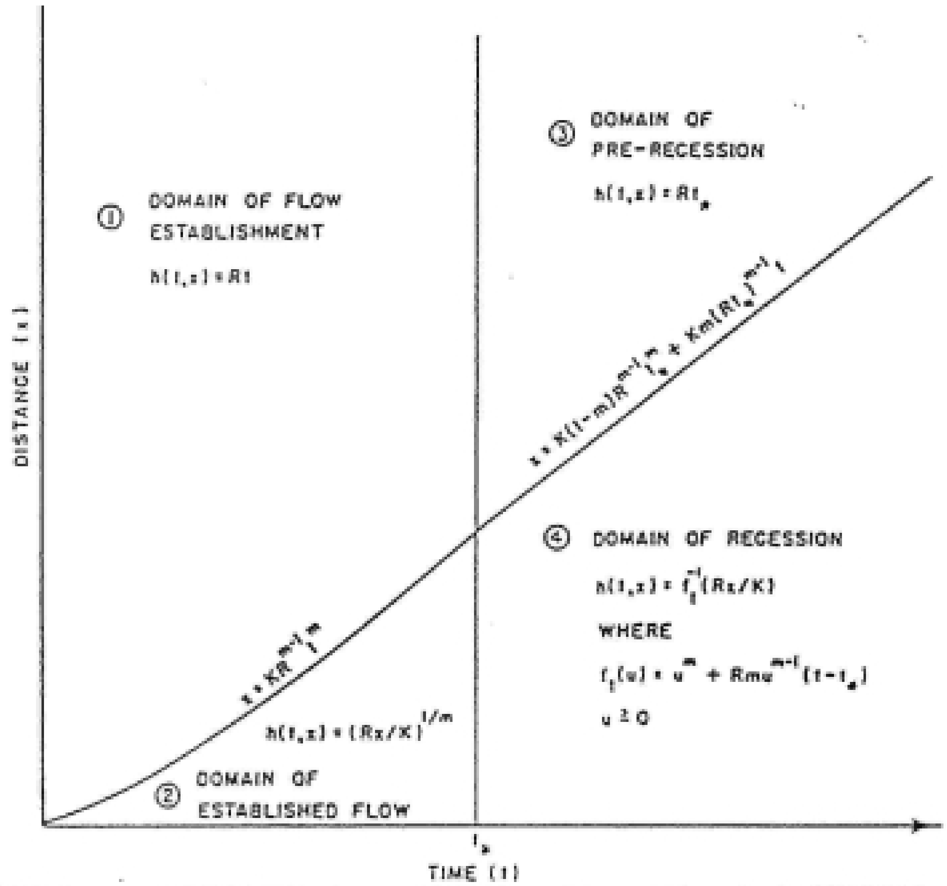

Figure 10.1. Domains in the t – x plane for solutions of the kinematic overland flow equations for a constant and uniform rainfall excess rate of duration t_*

$$0 \le t \le t_*$$

and (A15)

$$x \ge KR^{m-1}t^m$$

b. *Domain 2*. This region of the plane represents time from zero until cessation of rainfall excess and distance down the plane such that concentration has not reached steady state, but flow has:

$$\begin{aligned} &0 \le t \le t_* \\ &Km^{-m}R^{m-1}t^m \le x \le KR^{m-1}t^m \end{aligned} \tag{A16}$$

c. *Domain 3*. This region of the plane represents time from zero until cessation of rainfall excess and distance down the plane such that concentration and flow have reached steady state:

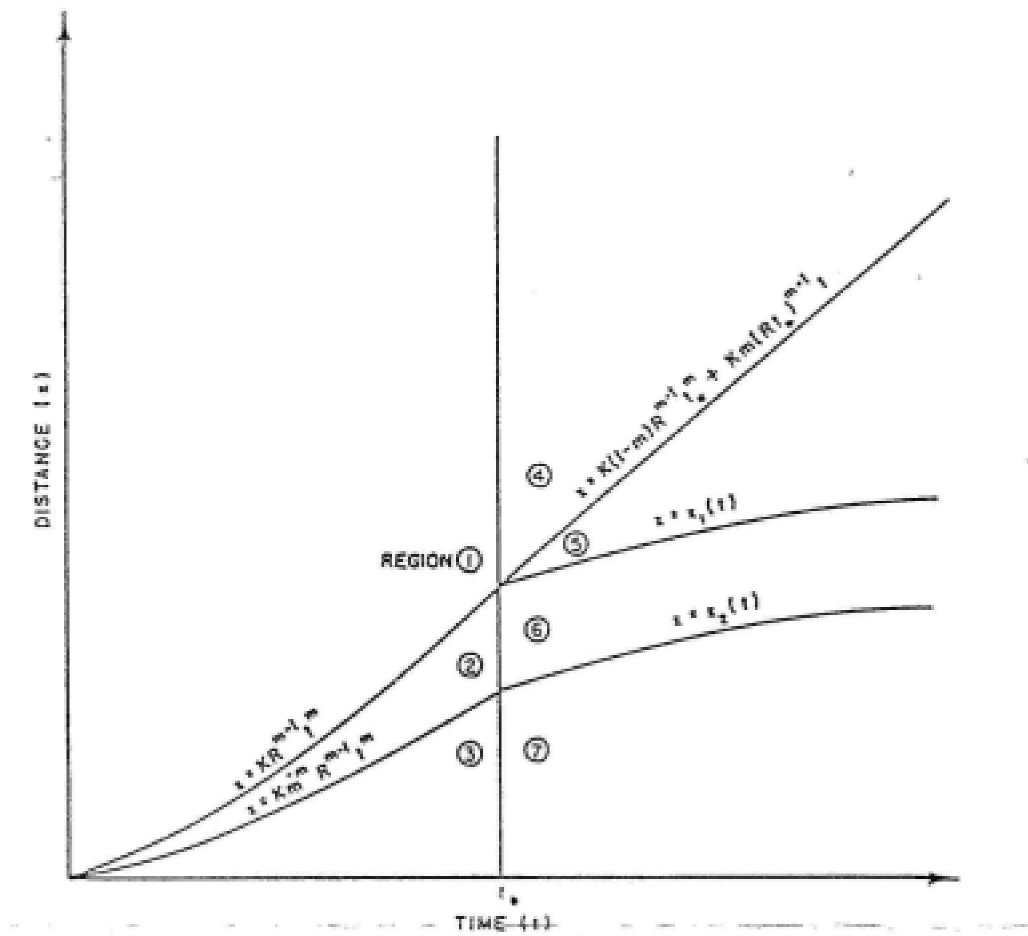

Figure 10.2. Domains in the t – x plane for solutions of the overland flow erosion equations for a constant and uniform rainfall excess rate of duration t_*

$$0 \le t \le t_*$$
$$0 \le x \le Km^{-m}R^{m-1}t^m \qquad \text{(A17)}$$

d. *Domain 4*. This region, corresponding to the domain of prerecession for flow, represents time after cessation of rainfall excess before depth of flow is receding, and before the arrival of the slower travelling concentration disturbance from the interaction of the water wave with cessation of rainfall excess:

$$t \ge t_*$$
$$x \ge K(1-m)R^{m-1}t_*^m + Km(Rt_*)^{m-1}t \qquad \text{(A18)}$$

In Domains 5-7, let

$$a(u) = t_* + (K_0u^m - mu^{-1}/(m+1))/R(m-1)\ , \qquad \text{(A19)}$$

and

$$b(u) = K(mK_0 u - u^{-m}/(m+1))/R(m-1) \tag{A20}$$

e. *Domain 5*. This region represents that portion of the domain of recession before the arrival of the concentration disturbance propagating from the interaction of the water wave with cessation of rainfall excess. With the above definitions of *a* and *b*, let

$$x_1(t) = b(a^{-1}(t)) \tag{A21}$$

where

$$K_0 = m(Rt_*)^{M+1}/(m+1) \tag{A22}$$

Finally, the region is defined as:

$$t \le t_*$$

and (A23)

$$x_1(t) \le x \le K(1-m)R^{m-1}t_* m + Km(Rt_*)^{m-1}t$$

f. *Domain 6*. This region represents that portion of the domain of recession after the arrival of the concentration disturbance propagating from the interaction of the water wave with cessation of rainfall excess and before the arrival of the concentration disturbance propagating from the upper boundary. With the above functions *a* and *b*, let

$$x_2(t) = b(a^{-1}(t)) \tag{A24}$$

where

$$K_0 = m(Rt_*/m)^{m+1}/(m+1) \tag{A25}$$

With these definitions, the region is bounded by:

$$t \ge t_*$$

and (A26)

$$x_2(t) \le x \le x_1(t)$$

g. *Domain 7*. This region represents that portion of the domain of recession after the arrival of the concentration disturbance propagating from the upper boundary:

$$t \ge t_*$$

and (A27)

$$0 \le x \le x_2(t)$$

2. Solutions in the Regions

a. *Domain 1*. In this region

$$c(t,x) = K_I + K_R(B/K - K_I)uF(u) \tag{A28}$$

where

$$u = KR^{m-1}t^m / m \tag{A29}$$

and

$$F(u) = \int_0^1 v^{1/m} \exp(K_R u(v-1))\mathrm{d}v \tag{A30}$$

b. *Domain 2*. In this region

$$\begin{aligned} c(t,x) = B/K + (K_I - B/K)(1 - \exp(K_R(x_0 - x)) \\ ((K_R x_0)^2 / mF(x_0/m) + 1 - K_I x_0))/(K_R x) \end{aligned} \tag{A31}$$

where

$$x_0 = KR^{m-1}((m(Rx/K)^{1/m} - Rt)/R(m-1)^m \tag{A32}$$

c. *Domain 3*. In this region

$$c(t,x) = B/K + (K_I - B/K)(1 - \exp(-K_R x))/K_R x \tag{A33}$$

d. *Domain 4*. In this region

$$c(t,x) = B/K + (K_I - B/K)(1 - K_R x_* / mF(x_*/m))\exp((K_R x_*/t_*)(t_* - t)) \tag{A34}$$

where

$$x_* = KR^{m-1}t_*^{\,m} \tag{A35}$$

e. *Domain 5*. In this region

$$c(t,x) = B/K + c_0 \exp(-K_R x) \tag{A36}$$

where

$$K_0 = (R(m-1)(t - t_*) + mh(t,x)/(m+1))h^m(t,x) \tag{A37}$$

and

$$c_0 = (c(a(1/Rt_*), b(1/Rt_*)) - B/K)\exp(K_R b(1/Rt_*)) \tag{A38}$$

where *c* is computed using the formula from domain 4, equation A34.

f. *Domain 6*. In this region

$$c(t,x) = B/K + c_0 \exp(-K_R(x - x_0)) \tag{A39}$$

where K_0 is defined above, $x_0 = K((m+1)K_0/m)^{m/(m+1)}R$, and

$$c_0 = c(t_*, x_0) - B/K \tag{A40}$$

with $c(t_*, x_0)$ computed using the formula for domain 2, equation A31.

g. *Domain 7*. In this region

$$c(t,x) = B/K + c_0 \exp(-K_R(x - x_0)) \tag{A41}$$

where K_0 and x_0 are defined above and

$$c_0 = c(t_*, x_0) - B/K \tag{A42}$$

with $c(t_*, x_0)$ computed using the formula for domain 3, equation A33.

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